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Nuclear Vibrations and Rotations of Like Nucleons in the Same Shell in Even–Even Nuclei

Ahmed Osman[1] **and M. A. Allam**[1]

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The vibrational and rotational motions in even nuclei are considered. A microscopic study of these motions leads to a relation between the vibrational motion in spherical nuclei and the rotational motion in deformed nuclei. Nuclei with like nucleons in the same shell are considered. The quadrupole two-body interactions are used in the large single j-shell of even nuclei. The energies and transition operators of nuclei in the nuclear rotational region are calculated using this microscopic method. Quadrupole moments are also calculated. These calculations are compared with the rotational model of the aligned coupling scheme. The present calculations are in good agreement with previous calculations.

1. INTRODUCTION

Studying the problems of like nucleons in one shell is of great help in providing a better understanding of the correlation between the nucleons. Exact solutions of very simple models have been compared with the solutions by existing approximate methods. This comparison makes it possible to gain an improved understanding of the approximate methods. In addition, this comparison serves as a tool for finding correlations to these approximate methods. In nuclear physics the long-range component of the residual force is responsible for the tendency to deformation, while the short-range component is responsible for giving a spherical system. In the last few years, promising progress has been made by regarding the collective model as a consequence of the interplay between these two tendencies, and by recognizing the appropriate coupling scheme for these two components (Mottleson, 1960; Bohr and Mottleson, 1962/1963). The aligned coupling scheme is used for the long-range component, while generalized seniority is introduced for the short-range component.

[1]Physics Department. Faculty of Science, Cairo University, Cairo, Egypt.

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Appropriate calculations with exact diagonalization of the Hamiltonian have been used in treating even nuclei with many particles outside a closed shell. There are two methods for these calculations, which are basic variants of the appropriate second quantization method. The first method is the Tam (1945)-Dancoff (1950) (TD) method and the second is the random-phase approximation (RPA) method. In the TD method the quasiparticle interaction is taken into account in the excited states, but that interaction does not affect the ground state. Therefore, the ground state of an even–even nucleus is the quasiparticle vacuum. The main shortcoming of the TD method is the asymmetric treatment of the ground and excited states. This defect has been corrected in the RPA method by including the quasiparticle interactions in all of the states. The approximate RPA method deviates from the shell model by an unsatisfactory description of spherical nuclei. The deviations between the approximate calculations and the shell model define two collective effects. One of these effects appears near the end of the shells, defining the spherical nuclei with a tendency for vibrational motion. The other effect appears near the middle of the shells, where the nuclei are deformed, with a tendency for rotational motion.

For low-lying levels in even–even nuclei, such as $Xe^{122-132}$, it was observed (Marinaga and Lark, 1965) that high anharmonicity in the vibration appears if the lower energy levels are assumed to be vibrational. Comparing the position of the 4^+ levels relative to those of the second 2^+ levels, one confirms this anharmonicity, noticing that in all known cases the 4^+ levels are slightly higher than the 2^+ levels. This is in contrast with the position and also the trend in the relative position with respect to the neutron number of the second 2^+ levels in permanently deformed osmium isotopes, in which the 2^+ levels are the first member of the gamma vibrational levels. This shows that the states observed in the spectra of even–even nuclei such as Xe are basically vibrational. However, the anharmonicity in the vibration cannot be ignored, and in addition the ground states of these nuclei are not permanently deformed. This effect of anharmonicity in vibration introduces the suggestion of the existence of quasirotational spectra as an intermediate situation between the vibrational motion in spherical nuclei and the rotational motion in deformed nuclei.

One of the most interesting methods of studying the vibrational motion is the random-phase approximation (Marumori et al., 1968), which gives the equation of motion of the functions of the two-body correlations and the quadrupole correlations in a harmonic approximation. On the other hand, the rotational motion is considered (Bohr and Mottleson, 1975) basically by using the cranking model, by introducing a microscopic basis (Bohr, 1977) to explain some properties of the rotating nuclei. From these studies it was found that one of the most important characteristics for the rotational phenomena in nuclei is the strong quadrupole two-body correlations. Thus, an approach was needed that describes both vibrational and rotational motions on the basis of both quantum mechanical and microscopic descriptions.

In the present work, we consider the vibrational and rotational motions in even–even nuclei on a microscopic basis. It is our aim to obtain a relation between the vibrational motion in spherical nuclei and the rotational motion in deformed nuclei. We use the quadrupole two-body interactions in the large single j-shell for even nuclei. With this approach, we calculate the energies, transition operators, and quadrupole moments, in order to compare these calculated values with previously calculated values.

In Section 2, the mathematical formulas and expressions are introduced. Numerical calculations and results are presented in Section 3. Section 4 is devoted to discussion and calculations.

2. MATHEMATICAL FORMULAS AND EXPRESSIONS

Marumori et al. (1968) suggested a method for calculating the vibrational and rotational motions in even nuclei based on adopting the single j-shell with nucleons interacting through the pairing plus the quadrupole force. In the present work, we shall use only a pure quadrupole force for the cases considered here.

We introduce a brief consideration of this microscopic theory, keeping in mind that we shall use the obtained results for the special cases we consider in the present work. To construct the algebra of the problem, let us define the conventional pair operators

$$A^+_{JM} = (1/\sqrt{2}) \sum_{m_1 m_2} \langle jjm_1m_2/JM\rangle C^+_{jm_1} C^+_{jm_2} \tag{1}$$

$$B^+_{JM} = \sum_{m_1 m_2} \langle jjm_1m_2/JM\rangle C^+_{jm_1} (-)^{j+m_2} C_{j-m_2} \tag{2}$$

which satisfy the relations

$$A^+_{JM} = (-)^J A_{JM} \tag{3}$$

$$B^+_{JM} = (-)^M B_{J-M} \tag{4}$$

From equation (3), we see that J is only an even integer, so that $J = 0, 2, 4, \ldots$. Then, the Hamiltonian including both the pairing plus the quadrupole forces is given as

$$H = \varepsilon_0 N - \tfrac{1}{2} G P^+_{00} - \tfrac{1}{2}\chi \sum_M Q^+_{2M} Q_{2M} \tag{5}$$

$$N = (2j+1)^{1/2} B^+_{00}, \qquad P^+_{00} = (2j+1)^{1/2} A^+_{00}, \qquad Q^+_{2M} = qB^+_{2M} \tag{6}$$

In equation (5), G and χ denote the strengths of the pairing forces and the quadrupole force, respectively. Here ε_0 and P_{00}^+ are the single-particle energy and the pairing quasiparticle operator, respectively. The operators N and Q_{2M} are the number and mass quadrupole moment operators, respectively. The quantity q is the reduced matrix element of the single-particle quadrupole moment and is defined by

$$\langle jm_1/r^2 Y_{2M}(\theta, \phi)/jm_2\rangle = q(-)^{j+m_2}\langle jjm_1m_2/2M\rangle \tag{7}$$

The commutation relations among the pair operators A_{JM}, A_{JM}^+, B_{JM}, and B_{JM}^+ are given as

$$\begin{aligned}[A_{J_1M_1}, A_{J_2M_2}^+] = {} & \tfrac{1}{2}\{1+(-)^{J_1}\}\,\delta_{J_1J_2}\delta_{M_1M_2} \\ & -\tfrac{1}{2}\sum_{J_3M_3}\{1+(-)^{J_1}\}\{1+(-1)^{J_2}\} \\ & \times Z(J_1J_2J_3)\langle J_2J_3M_2M_3/J_1M_1\rangle B_{J_3M_3}\end{aligned} \tag{8}$$

$$\begin{aligned}[B_{J_3M_3}, A_{J_1M_1}^+] = {} & \tfrac{1}{2}\sum_{J_2M_2}\{1+(-)^{J_1}\}\{1+(-)^{J_2}\} \\ & \times Z(J_1J_2J_3)\langle J_2J_3M_2M_3/J_1M_1\rangle A_{J_2M_2}^+\end{aligned} \tag{9}$$

$$\begin{aligned}[B_{J_1M_1}, B_{J_3M_3}^+] = {} & \sum_{J_2M_2}\{1-(-)^{J_1+J_2+J_3}\} \\ & \times Z(J_1J_2J_3)\langle J_2J_3M_2M_3/J_1M_1\rangle B_{J_2M_2}\end{aligned} \tag{10}$$

where

$$Z(J_1J_2J_3) \equiv [(2J_2+1)(2J_3+1)]^{1/2}W(jjJ_3J_2, J_1j) \tag{11}$$

The commutation relations given by equations (8)-(10) are regarded as fundamental, and replace the anticommutation relations among the one-particle operators C_{jm} and C_{jm}^+ as

$$\{C_{jm_1}^+, C_{jm_2}\}_+ = \delta_{m_1}\delta_{m_2}, \qquad \{C_{jm_1}^+, C_{jm_2}\}_+ = 0 \tag{12}$$

From the previous equations the definition of N is given by

$$N = \sum_m C_{jm}^+ C_{jm} \tag{13}$$

Instead of the definition given by equation (13) for N, the following relation is used:

$$\hat{N} = \tfrac{1}{2}\left\{\sum_M 2A_{JM}^+A_{JM} + 2A_{JM}A_{JM}^+ + B_{JM}^+B_{JM} + B_{JM}B_{JM}^+ \right. \\ \left. -\sum_J (2J+1)[1+(-)^J]\right\} \tag{14}$$

which is independent of equations (8)-(10).

From these equations we see that the basic idea in the Marumori work is to neglect the composite nature of the pair operators defined in equations

(1) and (2). Also, he regarded the pair operators A^+_{JM} and B^+_{JM} as fundamental operators that must satisfy equations (3), (4), (8)-(10), and (14). Then, he solved the dynamics of the system, using the Hamiltonian given by equation (5), which consists of pair operators. It is notable that if we restrict ourselves to the pair operators A^+_{JM} and B^+_{JM} with $J=0$, then equations (8)-(10) are reduced to the commutation relations characterizing the quasiparticle formalism for the pairing correlations (Kerman, 1961). Also, if we restrict our attention to the pair operators B^+_{JM} with $J\leq 2$, equation (10) reduces to the commutation relations characterizing the Elliot (1958) model. Keeping in mind that $Q^+_{2M}=qB^+_{2M}$, then we can relate the total angular momentum J_K introduced by Marumori et al. (1967) to the operators B^+_{1K} through the relation

$$J_K=[\tfrac{1}{3}j(j+1)(2j+1)]^{1/2}B^+_{1K} \tag{15}$$

This method is useful for forming a bridge between the vibrational motion in spherical nuclei and the rotational motion in deformed nuclei. For that, we do not need to define any free ground state such as a spherical or deformed Hartree-Fock ground state at the outset. Furthermore, it would be a more suitable method to investigate strong two-body correlations resulting from the pairing force and the quadrupole force. This happens because we did not employ the special technique of the generalized Hartree-Fock factorization proposed by Kerman and Klein (1963, 1965).

In the following, we write the equations of motion for the pair operators. We consider a special case by considering only pure quadrupole vibrations. We start with the general equations, by giving the following set of equations of motion:

$$\begin{aligned}
&[H, A^+_{JM}]\\
&\quad=\{\varepsilon^+-G[1+(-)^J]-\tfrac{1}{2}G(2j+1)\delta_{J0}\delta_{M0}\}A^+_{JM}\\
&\qquad+\tfrac{1}{2}G(2j+1)^{1/2}[1+(-)^J]B^+_{JM}A^+_{00}\\
&\qquad-\tfrac{1}{2}\chi q^2\sum_I[1+(-)^J][1+(-)^I][5(2I+1)]^{1/2}W(jjI2,Jj)\\
&\qquad\times\sum_{\Lambda K}\langle 2I\Lambda K/JM\rangle B^+_{2\Lambda}A^+_{IK}
\end{aligned} \tag{16}$$

$$\begin{aligned}
&[H,(-)^{J-M}A_{J-M}]\\
&\quad=-[\varepsilon^- -\tfrac{1}{2}G(2j+1)\delta_{J0}\delta_{M0}](-)^{J-M}A_{J-M}\\
&\qquad-\tfrac{1}{2}G(2j+1)^{1/2}[1+(-)^J]B^+_{JM}A_{00}\\
&\qquad+\tfrac{1}{2}\chi q^2\sum_I[1+(-)^J][1+(-)^I][5(2I+1)]^{1/2}W(jjI2,Jj)\\
&\qquad\times\sum_{\Lambda K}\langle 2I\Lambda K/JM\rangle B^+_{2\Lambda}(-)^{I-K}A_{I-K}
\end{aligned} \tag{17}$$

$$[H, B^+_{JM}]$$
$$= G(2j+1)^{1/2}\delta_{J0}\delta_{M0} - G[1+(-)^J]B^+_{JM}$$
$$+\tfrac{1}{2}G(2j+1)^{1/2}[1+(-)^J][A^+_{JM}A_{00} - (-)^{J-M}A_{J-M}A^+_{00}]$$
$$+\tfrac{1}{2}\chi q^2 \sum_I [1-(-)^{J+I}][5(2I+1)]^{1/2} W(jjI2, Jj)$$
$$\times \sum_{\Lambda K} \langle 2I\Lambda K/JM\rangle (B^+_{Ik}B^+_2 + B^+_2 B^+_{IK}) \tag{18}$$

$$\varepsilon^+ = \varepsilon^0[1+(-)^J] \pm \tfrac{1}{2}\chi q^2 \sum_I [1+(-)^J][1+(-)^I]5(2I+1) W(jjI2; Jj) \tag{19}$$

Let $|N; \gamma LL_2\rangle$ represent one of the members of the rotational band built on the ground state of the N-particle system. The quantities L and L_Z are the quantum numbers of the angular momentum and its projection, respectively. γ refers to a set of additional quantum numbers specifying the rotation. We keep in mind that in our case N is an even number and so L is an even integer, $L = 0, 2, 4, \ldots$. We then get the equations

$$\{W_N^{(L)} - W_{N-2}^{(L')}\}\langle N; \gamma LL_z|A^+_{JM}|N-2; \gamma L'L'_z\rangle$$
$$= \langle N; \gamma LL_z|[H, A^+_{JM}]|N-2; \gamma L'L'_z\rangle \tag{20}$$
$$\{W_N^{(L)} - W_{N-2}^{(L')}\}\langle N; \gamma LL_z|(-)^{J-M}A_{J-M}|N+2; \gamma L'L'_z\rangle$$
$$= \langle N; \gamma LL_z|[H, (-)^{J-M}A_{J-M}]|N+2; \gamma L'L'_z\rangle \tag{21}$$
$$\{W_N^{(L)} - W_N^{(L')}\}\langle N; \gamma LL_z|B^+_{JM}|N; \gamma L'L'_z\rangle$$
$$= \langle N; \gamma LL_z|[H; B^+_{JM}]|N; \gamma L'L'_z\rangle \tag{22}$$

where $W_N^{(L)}$ is the energy eigenvalue of the state $|N; \gamma LL_2\rangle$ subject to the definition

$$W_N^{(LL')} = W_N^{(L)} - W_N^{(L')}, \qquad \lambda^{L'}_{N,N-2} = (W_N^{(L')} - W_{N-2}^{(L')})/2 \tag{23}$$

Then equations (20)-(22) can be written as

$$W_N^{(L'L')}\langle N; \gamma LL_Z|A^+_{JM}|N-2; \gamma L'L'_Z\rangle$$
$$= \langle N; \gamma LL_Z|[H - \lambda^{(L')}_{N,N-2}\cdot \hat{N}, A^+_{JM}]|N-2; \gamma L'L'_Z\rangle \tag{24}$$
$$W_N^{(LL')}\langle N; \gamma LL_Z|(-)^{J-M}A_{J-M}N-2; \gamma L'L'_Z\rangle$$
$$= \langle N; \gamma LL_Z|[H - \lambda^{(L')}_{N+2,N}\cdot \hat{N}, (-)^{J-M}A_{J-M}]|N+2; L'L'_Z\rangle \tag{25}$$
$$W_N^{(LL')}\langle N; \gamma LL_Z|B^+_{JM}|N; \gamma L'L'_Z\rangle$$
$$= \langle N; \gamma LL_Z|[H; B^+_{JM}]|N; \gamma L'L'_Z\rangle \tag{26}$$

Equations (24)-(26) are considered the fundamental equations for this approach.

In solving the dynamics of a system that consists of pair operators, a basic approximation is introduced into the theory. This basic approximation is introduced by considering that the amplitudes

$$\langle N;\ \gamma LL_Z|A^+_{JM}|N-2;\ \gamma L'L'_Z\rangle$$
$$\langle N;\ \gamma LL_Z|(-)^{J-M}A_{J-M}|N+2;\ \gamma L'L'_Z\rangle$$
$$\langle N;\ \gamma LL_Z|B^+_{JM}|N;\ \gamma L'L'_Z\rangle$$

for values of $J\leq 2$ and for the case $L'=L$ with $L-2\geq 0$ are very large compared with all other amplitudes of pair operators, so that the other amplitudes may be neglected. With that basic approximation, the fundamental equations (24)-(26) are reduced to a simple set of coupled equations:

$$W_N^{(LL')}\langle N;\ \gamma LL_Z|A^+_{2M}|N-2;\ \gamma L'L'_Z\rangle$$
$$=\langle N;\ \gamma LL_Z|[H-\lambda^{(L')}_{N,N-2}\cdot\hat{N},\ A^+_{2M}]|N-2,\ \gamma L'L'_Z\rangle \qquad (27)$$
$$W_N^{(LL')}\langle N;\ \gamma LL_Z|(-)^{2-M}A_{2-M}|N+2;\ \gamma L'L'_Z\rangle$$
$$=\langle N;\ \gamma LL_Z|[H-\lambda^{(L')}_{N+2,N}\cdot\hat{N},\ (-)^{2-M}A_{2-M}]|N+2,\ \gamma L'L'_Z\rangle \qquad (28)$$
$$W^{(LL')}\langle N;\ \gamma LL_Z|B^+_{2M}|N;\ \gamma L'L'_Z\rangle$$
$$=\langle N;\ \gamma LL_Z|[H,\ B^+_{2M}]|N;\ \gamma L'L'_Z\rangle \qquad (29)$$

where

$$L'=L-2\geq 0, \qquad L'=0,2,4,\ldots$$
$$\langle N;\ \gamma LL_Z|[H-\lambda^{(L')}_{N,N-2}\cdot\hat{N},\ A^+_{00}]|N-2;\ \gamma LL_Z\rangle=0 \qquad (30)$$
$$\langle N;\ \gamma LL_Z|[H-\lambda^{(L')}_{N+2,N}\cdot\hat{N},\ A_{00}]|N+2;\ \gamma LL_Z\rangle=0 \qquad (31)$$

keeping in mind that $A^+_{1M}=0$ and that $J=0,2,4,\ldots$ from the relation $A^+_{JM}=(-)^J A_{JM}$; $B^+_{1K}=[j(j+1)(2j+1)/3]^{-1/2}\cdot\hat{J}_k$; $B^+_{00}=(2j+1)^{-1/2}\hat{N}$; and also $[H,\hat{N}]=[H,\hat{J}_k]=[N,\hat{J}_k]=0$. Equations (27)-(29) are connected with excitations of the N-particle system with excitation energies $W_N^{(LL')}$. From these formulations, we notice that equations (30) and (31) could be simply reduced to the BCS equation, and this happens in the case that the quadrupole force is absent (for $\chi\to 0$) and also for $L=0$ and with the λ's regarded as the chemical potentials.

Then, we can introduce the reduced amplitudes through the relations

$$\langle N;\ \gamma LL_Z|A^+_{JM}|N-2;\ \gamma LL'_Z\rangle$$
$$=\langle L'JL'_ZM|LL_Z\rangle\langle N;\ \gamma L\|A^+_J\|N-2;\ \gamma L'\rangle \qquad (32)$$
$$\langle N;\ \gamma LL_Z|(-)^J A_{J-M}|N+2;\ \gamma L'L'_Z\rangle$$
$$=\langle L'JL'_ZM|LL_Z\rangle\langle N;\ \gamma L\|A_J\|N+2;\ \gamma L'\rangle \qquad (33)$$

$$\langle N;\, \gamma L L_Z | B^{+}_{JM} | N;\, \gamma L' L'_Z \rangle$$
$$= \langle L' J L'_Z M | L L_Z \rangle \langle N;\, \gamma L \| B^{+}_{J} \| N;\, \gamma L' \rangle \tag{34}$$

Therefore, by using equations (16)-(18) and introducing these in equations (27)-(29), we can write

$$W^{(LL')}_{N} \psi^{+}(N;\, LL') = 2\varepsilon^{+}_{N}(LL')\psi^{+}(N;\, LL')$$
$$+ 2^{1/2}[\Delta_{N,N-2}(L') + f_{N,N-2}(LL')]\psi^{0}(N;\, LL') \tag{35}$$

$$W^{(LL')}_{N} \psi^{-}(N;\, LL') = -2\varepsilon^{-}_{N}(LL')\psi^{-}(N;\, LL')$$
$$+ 2^{1/2}[\Delta_{N,N+2}(LL') + f_{N+2,N}(LL')]\psi^{0}(N;\, LL') \tag{36}$$

$$W^{(LL')}_{N} \psi^{0}(N;\, LL') = -2\varepsilon^{0}_{N}(LL')\psi^{0}(N;\, LL')$$
$$+ 2^{1/2}\Delta_{N,N-2}(L')\psi^{+}(N;\, LL')$$
$$- 2^{1/2}\Delta_{N+2,N}(L')\psi^{-}(N;\, LL') \tag{37}$$

with $L' = L - 2 \geq 0$ $(L' = 0, 2, 4, 6, \ldots)$.

In equations (35)-(37), $\psi^{+}(N;\, LL')$, $\psi^{-}(N;\, LL')$, and $\psi^{0}(N;\, LL')$ are the reduced amplitudes of the pair operators with $J = 2$. These amplitudes are defined as

$$\psi^{+}(N;\, LL') \equiv \langle N;\, \gamma L \| A^{+}_{2} \| N-2;\, \gamma L' \rangle \tag{38}$$

$$\psi^{-}(N;\, LL') \equiv \langle N;\, \gamma L \| A_{2} \| N+2;\, \gamma L' \rangle \tag{39}$$

$$\psi^{0}(N;\, LL') \equiv \langle N;\, \gamma L \| B^{+}_{2} \| N,\, \gamma L' \rangle \tag{39'}$$

with $L' = L - 2 \geq 0$ and with $\varepsilon^{+}_{N}(LL')$ and $\varepsilon^{-}_{N}(LL')$ corresponding to the renormalized single-particle energies due to the deformation.

Then we get

$$\langle N;\, \gamma L \| Q^{+}_{2} \| N;\, \gamma L \rangle = (q \langle N;\, \gamma L \| B^{+}_{2} \| N;\, \gamma L \rangle) \tag{40}$$

$$\varepsilon^{+}_{N}(LL') \equiv \tfrac{1}{2}\varepsilon^{+} - \lambda^{L'}_{N,N-2} - G - 5\chi q W(jj22;\, 2j)[5(2L+1)]^{1/2}$$
$$\times W(LL22;\, 2L')\langle N;\, \gamma L \| Q^{+}_{2} \| N;\, \gamma L \rangle \tag{41}$$

$$\varepsilon^{-}_{N}(LL') \equiv \tfrac{1}{2}\varepsilon^{-} - \lambda^{L'}_{N+2,N} - 5\chi q W(jj22;\, 2j)[5(2L+1)]^{1/2}$$
$$\times W(LL22;\, 2L')\langle N;\, \gamma L \| Q^{+}_{2} \| N;\, \gamma L' \rangle \tag{42}$$

where $\varepsilon^{\pm}$ is defined by equation (19).

The quantity $\varepsilon^{0}_{N}(LL')$ is defined by

$$\varepsilon^{0}_{N}(LL') = G - \tfrac{5}{2}q^{2} W^{2}(jj12;\, 2j) R_{LL'} \tag{43}$$

where $R_{LL'} = L(L+1) - L'(L'+1)$ and this is connected with the rotational motion through the second term.

The gap $\Delta_{N,N-2}(L')$ and the gap $f_{N,N-2}(LL')$ are defined as

$$2^{1/2}\Delta_{N,N-2}(L') \equiv G(2j+1)^{1/2}\langle N;\, \gamma L'\| A_0^+ \| N-2;\, \gamma L'\rangle \tag{44}$$

$$2^{1/2}\Delta_{N+2,N}(L') \equiv G(2j+1)^{1/2}\langle N;\, \gamma L'\| A_0^- \| N+2;\, \gamma L'\rangle \tag{44'}$$

$$\begin{aligned} 2^{1/2} f_{N,N-2}(LL') \equiv -2\chi q^2 \sum_{I=0,2} & [5(2I+1)]^{1/2} W(jjI2;\, 2j) \\ & \times [5(2L'+1)]^{1/2} W(LL'22;\, IL) \\ & \times \langle N;\, \gamma L'\| A_I^+ \| N-2;\, \gamma L'\rangle \end{aligned} \tag{45}$$

Equations (35)-(37) show the advantage of the arrangement of pair operators in the quadratic terms on the right-hand side of equations (16)-(18), which becomes more clear. Our present basic approximation leads to equations (35)-(37), which are written down in a compact closed form with respect to $\psi^{(\pm)}(N;\, LL')$ and $\psi^{(0)}(N;\, LL')$ and do not contain amplitudes such as $\psi^{(\pm)}(N\pm 2;\, LL')$ explicitly. Thus we can use equations (27)-(31) together with matrix elements of equations (8)-(11) (with respect to the states under consideration) to determine all the amplitudes of the states and their normalizations and eigenvalues $\omega_N^{(LL')}$, $\lambda_{N,N+2}^{(L')}$, and $\lambda_{N,N-2}^{(L')}$ completely.

Let us introduce an approximate formal solution of the excitation energy, in order to show the physical meaning of the solution. For this purpose equations (35)-(37) can be rewritten as

$$\begin{aligned} &(2\varepsilon_N^+(LL') - \omega_N^{(LL')})\psi^{(+)}(N;\, LL') \\ &\quad + 2^{1/2}\{\Delta_{N,N-2}(L') + f_{N,N-2}(LL')\}\psi^{(0)}(N;\, LL') = 0 \end{aligned} \tag{46}$$

$$\begin{aligned} &(-2\varepsilon_N^-(LL') - \omega_N^{(LL')})\psi^{(-)}(N;\, LL') \\ &\quad + 2^{1/2}\{\Delta_{N,N+2}(L') + f_{N+2,N}(LL')\}\psi^{(0)}(N;\, LL') = 0 \end{aligned} \tag{47}$$

$$\begin{aligned} &(-2\varepsilon_N^0(LL') - \omega_N^{(LL')})\psi^{(0)}(N;\, LL') \\ &\quad + 2^{1/2}\Delta_{N,N-2}(L')\psi^{(+)}(N;\, LL') - 2^{1/2}\Delta_{N+2,N}(L')\psi^{(-)}(N;\, LL') = 0 \end{aligned} \tag{48}$$

Equations (46)-(48) are subject to the condition

$$\begin{vmatrix} (2\varepsilon_N^+(LL') - \omega_N^{(LL')}) & 0 & (2^{1/2}\{\Delta_{N,N-2}(L') + f_{N,N-2}(LL')\}) \\ 0 & (-2\varepsilon_N^{(-)}(LL') - \omega_N^{(LL')}) & (-2^{1/2}\{\Delta_{N+2,N}(L') + f_{N+2,N}(LL')\}) \\ (2^{1/2}\Delta_{N,N-2}(L')) & (-2^{1/2}\Delta_{N+2,N}(L')) & (-2\varepsilon_N^0(LL') - \omega_N^{(LL')}) \end{vmatrix} = 0 \tag{49}$$

Equation (49) is not simply a cubic equation with respect to $\omega_N^{(LL')}$, because the coefficients (which should be determined self-consistently) are generally implicit functions of $\omega_N^{(LL')}$. Since, by definition, we have the condition

$$\omega_N^{(LL')} = -\omega_N^{(LL')}$$

we are led to another condition given by the expression

$$\begin{vmatrix} (2\varepsilon_N^{(+)}(L'L)+\omega_N^{(LL')}) & 0 & (2^{1/2}\{\Delta_{N,N-2}(L)+f_{N,N-2}(L'L)\}) \\ 0 & (-2\varepsilon_N^{(-)}(L'L)+\omega_N^{(LL')}) & (-2^{1/2}\{\Delta_{N+2,N}(L)+f_{N+2,N}(L'L)\}) \\ (2^{1/2}\Delta_{N,N-2}(L)) & (-2^{1/2}\Delta_{N+2,N}(L)) & (-2\varepsilon_N^{(0)}(L'L)+\omega_N^{(LL')}) \end{vmatrix} = 0 \quad (50)$$

From equations (49) and (50), we have the following relation (in which the coefficients are of the symmetrized form with respect to L and L'):

$$(\omega_N^{(LL')})^3 - C_N^{(2)}(LL')(\omega_N^{(LL')})^2 - C_N^{(1)}(LL')\omega_N^{(LL')} - C_N^{(0)}(LL') = 0 \quad (51)$$

where

$$\begin{aligned} C_N^{(2)}(LL') &= -C_N^{(2)}(L'L) \\ &\equiv [\varepsilon_N^{+}(LL') - \varepsilon_N^{(-)}(LL') - \varepsilon_N^{(0)}(LL') \\ &\quad - \varepsilon_N^{(+)}(L'L) + \varepsilon_N^{(-)}(L'L) + \varepsilon_N^{(0)}(L'L)] \end{aligned} \quad (52)$$

$$\begin{aligned} C_N^{(1)}(LL') &= C_N^{(1)}(L'L) \\ &\equiv 2\{\varepsilon_N^{(+)}(LL')\varepsilon_N^{(-)}(LL') + \varepsilon_N^{(+)}(L'L)\varepsilon_N^{(-)}(L'L) \\ &\quad + \varepsilon_N^{(0)}(LL')[\varepsilon_N^{(+)}(LL') - \varepsilon_N^{(-)}(LL')] \\ &\quad + \varepsilon_N^{(0)}(L'L)[\varepsilon_N^{(+)}(L'L) - \varepsilon_N^{(-)}(LL')]\} \\ &\quad + \{[\Delta_{N+2,N}(L') + f_{N+2,N}(LL')]\Delta_{N+2,N}(L')\Delta_{N+2,N}(L') \\ &\quad + [\Delta_{N,N-2}(L') + f_{N,N-2}(LL')]\Delta_{N,N-2}(L') \\ &\quad + [\Delta_{N+2,N}(L) + f_{N+2,N}(L'L)]\Delta_{N+2,N}(L) \\ &\quad + [\Delta_{N,N-2}(L) + f_{N,N-2}(L'L)]\Delta_{N,N-2}(L)\} \end{aligned} \quad (53)$$

$$\begin{aligned} C_N^{(0)}(LL') &= -C_N^{(0)}(L'L) \\ &\equiv 2[2\varepsilon_N^{(+)}(LL')\varepsilon_N^{(-)}(LL')\varepsilon_N^{(0)}(LL') \\ &\quad - \varepsilon_N^{(+)}(LL')\varepsilon_N^{(-)}(L'L)\varepsilon_N^{(0)}(LL')] \\ &\quad + [\Delta_{N,N-2}(L') + f_{N,N-2}(L'L)]\Delta_{N,N-2}(L')\varepsilon_N^{(-)}(LL') \\ &\quad - [\Delta_{N,N-2}(L) + f_{N,N-2}(L'L)]\Delta_{N,N-2}(L')\varepsilon_N^{(-)}(L'L) \\ &\quad - [\Delta_{N+2,N}(L') + f_{N+2,N}(LL')]\Delta_{N+2,N}(L')\varepsilon_N^{(+)}(LL') \\ &\quad + [\Delta_{N+2,N}(L) + f_{N+2,N}(L'L)]\Delta_{N+2,N}(L)\varepsilon_N^{(+)}(L'L) \end{aligned} \quad (54)$$

Let us consider the following approximations:

$$\lambda_{N,N-2}^{(L)} - \lambda_{N+2,N}^{(L)} = \lambda_{N,N-2}^{(L')} - \lambda_{N+2,N}^{(L')} \quad (55)$$

$$C_N^{(0)}(LL') \equiv -C_N^{(0)}(L'L) = 0 \quad (56)$$

These approximations can be understood by noticing that in the case of a pure pairing force (i.e., $\chi=0$), equations (55) and (56) hold exactly and are not approximations. With the condition given by equation (55), we have to understand that one of the solutions for $\omega_N^{(LL')}$ becomes zero. These conditions as given by equations (55) and (56), together with equations (30) and (31), should be used in determining $\lambda_{N,N-2}^{(L)}$ and $\lambda_{N+2,N}^{(L)}$. Introducing the conditions expressed by equations (55) and (56) into equation (51), we get

$$(\omega_N^{(LL')})^3 - C_N^{(2)}(LL')(\omega_N^{(LL')})^2 - C_N^{(1)}(LL')\omega_N^{(LL')} - C_N^{(0)}(LL') = 0$$

Therefore

$$(\omega_N^{(LL')})^2 - C_N^{(2)}(LL')\omega_N^{(LL')} - C_N^{(1)}(LL') = 0 \tag{57}$$

Making use of condition (55), we have

$$\begin{aligned} C_N^{(2)}(LL') &= -\varepsilon_N^{(0)}(LL') + C_N^{(0)}(L'L) \\ &= -5\chi q^2 W(jj12, 2j)R_{LL'} \end{aligned}$$

Therefore, equation (57) can be written in the form

$$(\omega_N^{(LL')})^2 - 5\chi q^2 W(jj12; 2j)R_{LL'}\omega_N^{(LL')} - C_N^{(1)}(LL') = 0 \tag{58}$$

Equation (58) is a quadratic equation in $\omega_N^{(LL')}$, which can be solved, giving the following solution for the excitation energies:

$$\begin{aligned} \omega^{(LL')} &= \tfrac{5}{2}\chi q^2 W^2(jj12, 2j)R_{LL'} \\ &\quad + \{[\tfrac{5}{2}\chi q^2 W(jj12, 2j)R_{LL'}]^2 + C_N^{(1)}(LL')\}^{1/2} \end{aligned} \tag{59}$$

with $L = L'+2$, where $L' = 0, 2, 4, 6, \ldots.$

Direct calculations of equation (59) for the case of pure pairing force in the limit $x \to 0$ give

$$\omega_N^{(LL')} = 2[(\varepsilon_0 - \lambda_N)^2 + \Delta_N^2(LL')]^{1/2} \tag{60}$$

with

$$\varepsilon_0 - \lambda = \varepsilon_0 - \lambda_{N+2,N}^{(L=0)} = \tfrac{1}{2}G(2j+1)\left(1 - \frac{2N}{2j+1}\right) \tag{61}$$

where

$$\begin{aligned} \Delta_N^2(LL') &= \frac{1}{4}\Delta_{N+2,N}^2(L) + \Delta_{N,N-2}^2(L) + \Delta_{N+2,N}^2(L') + \Delta_{N,N-2}^2(L') \\ &= \frac{1}{4}G^2(2j+1)^2\frac{N-L'}{2j+1}\left(1 - \frac{N+L'}{2j+1}\right) \end{aligned} \tag{62}$$

From equations (60)-(62), we notice that equation (60) corresponds to the two-quasiparticle energy with "backing effects," while equations (61) and

(62) show that equation (60) represents the same spectrum as does the exact solution.

On the other hand, in the case of a pure quadrupole force, in the limit $G \to 0$, which is the pure Elliot-type rotational spectrum, we have

$$\omega_N^{(LL')} = (h^2/2g_0)R_{LL'} \tag{63}$$

where

$$g_0 = (h^2/10)\chi q^2 W^2(jj12, 2j), \qquad R_{LL'} = L(L+1) - L'(L'+1)$$

It is clear that, under the same approximations as these underlying the conventional random-phase approximation, equation (59) is reduced to an equation that describes the "phonon" spectra. Thus, our approach provides a bridge between the vibrational motion and the rotational motion.

3. NUMERICAL CALCULATIONS AND RESULTS

For numerical calculations, let us use the pair operators A_{JM}^{+} and the particle-hole operators B_{JM}^{+} following the Marumori *et al.* formalism, defined as

$$A_{JM}^{+} = (1/\sqrt{2}) \sum_{m_1 m_2} \langle jm_1 jm_2 | JM \rangle C_{jm_1}^{+} C_{jm_2}^{+} \tag{64}$$

$$B_{JM}^{+} = \sum_{m_1 m_2} \langle jm_1 jm_2 | JM \rangle C_{jm_1}^{+} (-)^{j+m_2} C_{j-m_2} \tag{65}$$

In the present calculations, we restrict our results to the case of a single j-shell. Equations (64) and (65) give a coupled set of equations of motion of the different operators A_{JM}^{+} and B_{JM}^{+}. These equations of motion are given by equations (16)-(18). The typical rotational spectrum in the case of a pure quadrupole force, which is the limiting case when $G \to 0$, is given by

$$E_L = (1/2g)L(L+1) \tag{66}$$

The reduced transition probability for the electric quadrupole transitions E_2 within a rotational band is defined as

$$B(E2, L \to L') = \frac{1}{2L'+1} \sum_{\gamma, L_z, L'_z} |\langle N, \gamma L L_z | M_E(20) | N, \gamma L L'_z \rangle|^2$$

$$= \langle L020 | L'0 \rangle^2 \langle N, \gamma L \| M_E(20) \| N, \gamma L' \rangle \tag{67}$$

But the diagonal matrix element of the $M_E(20)$ operator is proportional to the intrinsic quadrupole moment Q_0,

$$\langle N, \gamma L \| M_E(20) \| N, \gamma L' \rangle = (5/16)^{1/2} e Q_0 \tag{68}$$

Then we can obtain a simple expression for the E_2 transition probabilities,

$$B(E2, L\to L') = (5/16)e^2 Q_0 \langle L020|L'0\rangle^2 \tag{69}$$

The static quadrupole moment is defined as the expectation value of the quadrupole operator,

$$eQ_0 = (16\pi/5)^{1/2} \int \rho(r) r^2 Y_{20}(\theta, \phi)\, dr \tag{70}$$

in the state $|N; \gamma LL_z\rangle$, where $L'_z = L_z$ (i.e., $M = K$) ($\gamma = L'_z$). This means that

$$Q_L = \langle N, \gamma LL_z|(1/e)Q_0|N, \gamma LL_z\rangle \tag{71}$$

Therefore,

$$Q_L = (16\pi/5)^{1/2}(1/e)\langle N; \gamma LL_z| \int dr\, \rho(r) r^2 Y_{20}(\theta, \phi)|N, \gamma LL_z\rangle \tag{72}$$

or

$$Q_L = \frac{3K^2 - L(L+1)}{(L+1)(2L+3)} Q_0 \tag{73}$$

For the case of $K = 0$, we get for the static quadrupole moment a simple expression,

$$Q_L = -\frac{L}{(2L+3)} Q_0 \tag{74}$$

In equations (66)-(74), g is the moment of inertia, L is the angular momentum, and Q_0 is the intrinsic quadrupole moment. In the present calculations of the transition probabilities and quadrupole moments, we have to extend the generalized sum rules introduced by Marumori et al. We have the equation

$$\langle N; \gamma L\| Q_2^+ \| N, \gamma L\rangle = q\langle N; \gamma L\| B_2^+ \| N; \gamma L\rangle \tag{75}$$

where q is given by the equation

$$\langle jm_1|r^2 Y_{2M}(\theta, \phi)|j - m_2\rangle = q(-)^{j+m_2}\langle jm_1 jm_2|2M\rangle \tag{76}$$

The Hamiltonian of pure quadrupole vibrations is given by the expression

$$H = -\tfrac{1}{2}\chi \sum_M \hat{Q}_M (-)^M \hat{Q}_{-M} = -\tfrac{1}{2}\chi \sum_M Q_{2M}^+ Q_{2M} \tag{77}$$

This expression is obtained easily as the limiting case when $G\to 0$, in the absence of the pairing force. In equation (77) χ is the strength of the interaction. The mass quadrupole moment $\hat{Q}_M$ is expressed using the particle-hole operator B_{2M}^+ as

$$\hat{Q}_M = qB_{2M}^+ \tag{78}$$

where q is the reduced matrix element of the single-particle quadrupole moment. This single-particle quadrupole moment is defined as

$$\langle jm_1|r^2 Y_{2M}(\theta, \phi)|j-m_2\rangle = q(-)^{j+m_2}\langle jm_1 jm_2|2M\rangle \tag{79}$$

Then the ground-state energy E_0 and $\langle H\rangle$ for the single-particle energies are given as

$$E_0 = \langle H\rangle = \frac{\langle N;\, \gamma LL_z|H|N;\, \gamma LL_z\rangle}{\langle N;\, \gamma LL_z|N;\, \gamma LL_z\rangle} \tag{80}$$

$$\langle H\rangle = \frac{\langle N;\, \gamma LL_z|-\frac{1}{2}\chi\, \Sigma_M\, q^2 B^+_{2M}(-)^M B_{2-M}|N;\, \gamma LL_z\rangle}{\langle N;\, \gamma LL_z|N;\, \gamma LL_z\rangle} \tag{81}$$

The state function $|N;\, \gamma LL_z\rangle$ is a normalized functions, which means that

$$\langle N;\, \gamma LL_z|N;\, \gamma LL_z\rangle = 1 \tag{82}$$

Then, in the case of quadrupole force for $G=0$, we get

$$\begin{aligned} E_0 = \langle H\rangle &= \langle N;\, \gamma LL_z|-\tfrac{1}{2}\chi q^2 B^+_{20}B_{20}|N;\, \gamma LL_z\rangle \\ &= -\tfrac{1}{2}\chi q^2\langle N;\, \gamma LL_z|B^+_{20}B_{20}|N;\, \gamma LL_z\rangle \end{aligned} \tag{83}$$

Then, by using the commutation relation given by equation (10), we get for the case of $J=2$ and $M=0$

$$B^+_{20}B_{20} = \left|\sum_{m_1 m_2}\langle jjm_1 m_2|20\rangle C^+_{jm} C_{j-m_2}\right|^2 \tag{84}$$

where

$$B^+_{20} = B_{20} = B^+_{20} \tag{85}$$

Thus,

$$\begin{aligned} \langle |B^+_{20}|^2\rangle &\approx \langle B^+_{20}\rangle^2 \\ &\approx \left\langle \sum_{m_1 m_2}\langle jjm_1 m_2|20\rangle C^+_{jm_1} C_{j-m_2}\right\rangle^2 \\ &\approx \left\langle \left|\sum_{m_1 m_2}\langle jjm_1 m_2|20\rangle C^+_{jm_1} C_{j-m_2}\right|^2\right\rangle \end{aligned} \tag{86}$$

or

$$\langle B^+_{20}\rangle^2 = \left|\sum_m f_m n_m\right|^2 \tag{87}$$

Therefore,

$$E_0 = \langle H\rangle = -\tfrac{1}{2}\chi q^2\langle B^+_{20}\rangle^2 \tag{88}$$

Thus, for the single-particle energy we have

$$\varepsilon_\lambda = -\chi q f_\lambda \langle B_{20}^{+} \rangle \tag{89}$$

where

$$\langle B_{20}^{+} \rangle = \sum_m f_m n_m \tag{90}$$

and n is the occupation number of the state $|jm\rangle$.

The factor F is given by

$$F_m = \frac{5^{1/2}[3m^2 - j(j+1)]}{[j(j+1)(2j+1)(2j-1)(2j+3)]^{1/2}} \tag{91}$$

From these equations we see that there are two prolate and oblate solutions according to the occupied and unoccupied states for the single-particle states with different values of m. This means that the total energy can be calculated as intrinsic states corresponding to the oblate and prolate shapes of the density distributions.

The energy difference between the ground states in the case of the oblate and prolate solutions is given by

$$E = -\frac{1}{2}\chi q^2 \frac{5}{16} \frac{(2j+1)^5}{j(j+1)(2j-1)(2j+3)} 3N^2(1-N^2)(1-2N) \tag{92}$$

where N is related to the particle number n as

$$N = n/(2j+1) \tag{93}$$

But, for nuclei with rotational motion, n has to satisfy the condition

$$0 \ll n \ll 2j+1 \tag{94}$$

Then, N satisfies the inequality

$$0 \ll N \ll 1 \tag{95}$$

Introducing the Hartree approximation by neglecting the exchange terms of the interaction, Thouless (1960) showed that the moment of inertia becomes equal to the ordinary moment of inertia calculated by using the Cranking model (Inglis, 1954). Then, the calculations show that the nuclei are expected to have rotational motions with oblate deformations in the mass region

$$0.304 \leq N \leq 0.414 \tag{96}$$

and with prolate deformations in the mass region

$$0.583 \leq N \leq 0.696 \tag{97}$$

Thus, the characteristics of the nuclear structure depend upon the limiting case of the pairing correlations, which is studied by Stephens (1975), quadrupole vibrations, and the rotations from quadrupole deformations (Neergard and Froundorf, 1976).

Numerical calculations were performed for different quantities using large values of j. These calculations were carried out at values of $N = 0.359$ and $N = 0.639$ where rotational motions are expected (Osman, 1987) with oblate and prolate deformations, respectively. The moment of inertia $\mathcal{J}$ and the intrinsic quadrupole moment Q_0 were calculated and the results compared with values obtained using the Hartree-Fock calculations. The results are shown in Table I. Calculations were also carried out for the quadrupole moment of inertia of the first 2^+ excited state Q_2, and for the reduced transition probabilities from the ground state to the first 2^+ state $B(E2, 0 \rightarrow 2)$. These calculations were done for a value of $N = 0.359$, for comparison with exact calculations given by Mulhall and Sîps (1964) for $j = 11/2$ and $n = 4$.

In Table I, the moment of inertia $\mathcal{J}$ is given in units of

$$j(j+1)(2j+1)/3\chi q^2$$

The intrinsic quadrupole moment is in units of

$$-4\pi q[(2j+1)/5]^{1/2}$$

The quadrupole moment of the first 2^+ excited state Q_2 in units of

$$-4\pi q^2/[j(j+1)(2j+1)(2j+1)(2j+3)]^{1/2}$$

and the reduced transition probabilities from the ground state to the first 2^+ excited state $B(E2, 0 \rightarrow 2)$ are in units of

$$5q^6/[j(j+1)(2j+1)(2j-1)(2j+3)]$$

$\mathcal{J}$ and Q_0 are calculated in the case of large j, and Q_2 and $B(E2, 0 \rightarrow 2)$ are calculated in the case of $j = 11/2$ and $n = 4$.

Table I. Calculated Moment of Inertia, Intrinsic Quadrupole Moments, Static Quadrupole Moments, and Reduced Transition Probabilities

	N	Present work	Previous result
$\mathcal{J}$	0.359	0.469	0.472 (Hartree calculation)
	0.639	0.469	0.472 (Hartree calculation)
Q_0	0.359	0.471	0.431 (Hartree calculation)
	0.639	0.471	0.431 (Hartree calculation)
Q_2	0.359	16.948	16.312 (Exact calculation)[a]
$B(E2; 0 \rightarrow 2)$	0.359	3784.639	3826.574 (Exact calculation)[a]

[a]See Marumori et al. (1967).

4. DISCUSSION AND CONCLUSION

In the present work, we studied the vibrational and rotational motions in even nuclei on a microscopic basis. We constructed a relation between the vibrational and rotational motions, in these nuclei. We calculated the energies, transition amplitudes, and quadrupole moments. From Table I, we see that the present calculations are in good agreement with previous calculations.

Thus, our present investigations bridge the vibrational and the rotational collective motions in nuclei. This is easily done by comparing the model of strong coupling (Khoo et al., 1976) of the quadrupole mode and the pair mode. The quadrupole mode plays an important role for both the vibrational and the rotational motions.

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